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Asymptotic pseudo-state stabilization of commensurate fractional-order nonlinear systems with additive disturbance

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Abstract The pseudo-state stabilization problem of commensurate fractional-order nonlinear systems is investigated. The concerned fractional-order nonlinear system is of parametric strict-feedback form with both unknown parameters and the additive disturbance. To solve this problem, a new nonlinear adaptive control law is constructed via fractional-order backstepping scheme. The developed fractional-order controller does not require the knowledge about both the interval of uncertain parameters and the upper bound of the additive disturbance. The asymptotic pseudo-state stability of the closed-loop system is proved in terms of fractional Lyapunov stability. Several examples are performed finally, and the efficiency is verified.

Keywords Fractional-order nonlinear system · Fractional Lyapunov stability · Adaptive control · Fractional-order backstepping

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1 Introduction

With the development of physics, fractional-order systems have been widely studied in dynamical systems and control. This is mainly due to the fact that many physical processes are well characterized by fractional-order differential equations [1–3]. For more details on the applications of fractional calculus, one can refer to the monographs [1–4] and the references therein. To distinguish with the state, the Lyapunov stability and the Lyapunov function in classical integer-order systems, in fractional-order systems they are called the pseudo-state, the fractional Lyapunov stability and the fractional Lyapunov function, respectively. You can refer to references [5–8] for distinguishing these concepts.

The pseudo-state stabilization problem of fractional-order systems has attracted the attention of many researchers recently. Most of the known results concentrated on the fractional-order linear systems. The early stability criterion is the Matignon theorem [9]. Then, the linear matrix inequality (LMI) representations were introduced in [1], and the sufficient and necessary conditions were investigated by [10–12] further. With respect to LMI conditions, the pseudo-state feedback stabilization of deterministic fractional-order linear systems was addressed in [5,13]. On the other hand, uncertainties are common phenomena in fractional-order systems. Therefore, many robust pseudo-state stabilization results were put forward recently, such as in [14] and the references therein. Besides, $\mathscr{H}_\infty$

control problems of fractional-order systems were proposed by [15–17], and fractional-order reference adaptive control was investigated in [18–20] and the references therein. However, real fractional-order systems are always nonlinear with uncertainties.

It is well known that Lyapunov direct method is the fundamental tool to stabilize nonlinear systems. An early fractional Lyapunov-like theory was investigated by [21] for fractional-order systems. In [22, 23], Mittag–Leffler stability and generalized Mittag-Leffler stability were introduced to describe fractional Lyapunov stability of fractional-order systems. Generalized Mittag-Leffler stability of multivariables fractional-order systems was investigated by [24] further. Trigeassou et al. [6] demonstrated that classical Lyapunov functions are valid for fractional-order systems. It should be emphasized that the fractional Lyapunov functions are introduced to describe the pseudo-state stability of fractional-order systems. However, finding appropriate fractional Lyapunov functions for fractional-order systems remains a tedious task. Some existing possible fractional Lyapunov functions for fractional-order systems can be found in [6,25,26].

As we know, a few results on the pseudo-state stabilization of fractional-order nonlinear systems have been reported in terms of fractional Lyapunov stability. In [27], the linear pseudo-state feedback was introduced to stabilize fractional-order nonlinear systems. Robust pseudo-state stabilization of fractional-order nonlinear complex networks was investigated by [28] via Lyapunov indirect approach. For some simple examples of fractional-order nonlinear systems pseudo-state stabilization, one can refer to [6,21–26]. Recently, fractional-order sliding mode control is well defined for pseudo-state stabilizing some specific fractional-order nonlinear systems, which one can refer to [29] and the references therein.

Inspired by the above results, we aim to investigate the pseudo-state stabilization problem of commensurate fractional-order nonlinear systems. As we know, backstepping is a well-known efficient methodology of stabilizing nonlinear systems, which has been widely applied in practical applications [30]. However, to our best knowledge, backstepping is restricted to the classical integer-order nonlinear systems. Besides the first example proposed in [2] and our previous theoretical results [31,32], there are few results on it. There are many works to be done with backstepping control laws design for fractional-order nonlinear systems. As the resulting control laws are with fractional-order forms, we call such methodology the fractional-order backstepping.

In our contributions, the pseudo-state stabilization problem of commensurate fractional-order nonlinear systems with both the parameter uncertainty and the additive disturbance is solved. By use of fractional-order backstepping scheme, the analytic form of pseudo-state feedback control laws of stabilizing uncertain fractional-order nonlinear systems is designed. The global convergence of the closed-loop system is guaranteed in terms of fractional Lyapunov stability. In our design, the uncertain system parameters are only assumed to be unknown constants. The additive disturbance is only required to be bounded by unknown upper bound, which is estimated by the designed adaptive law. Besides the estimate of the upper bound of the disturbance, the number of the system parameter estimates is equal to that of the unknown system parameters. The parameters in the designed control law are not related to the additive disturbance and unknown system parameters, which can be chosen freely for improving the performance of the closed-loop system. Before the main result, a general framework of adaptive fractional Lyapunov based design is well defined via control fractional Lyapunov function (CFLF) for fractional-order systems. Finally, several examples demonstrate the efficiency of the proposed control scheme.

The rest of the paper is organized as follows. In Sect. 2, some preliminaries are introduced. Our main results are presented in Sect. 3, and the analytic form of the control law is presented. The efficiency of the proposed pseudo-state control law is verified in several examples in Sect. 4. Finally, some conclusions are summarized in Sect. 5.

We use the following notations. The real number and n dimension real space are $\mathbb{R}$ and $\mathbb{R}^n$, respectively. The transpose of a matrix A is denoted by $A^\top$. $\|\cdot\|$,$|\cdot|$ denote the norm and the abstract, respectively. $\lceil\cdot\rceil$ is the ceiling function. A matrix $A > 0$ means that A is a positive definite matrix. The symbol D^ν shorted for D_t^ν, where t is the time, represents the fractional-order derivative operator with Caputo type.

2 Preliminaries

In this paper, the Caputo fractional-order derivative is used.

Definition 1 [4] Let $f(t)$ is a real continuously differentiable function. The Caputo fractional-order derivative with order $0 < \nu < 1$ on $t > 0$ is defined by

$$D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}} \mathrm{d}\tau, \tag{1}$$

where $n = \lceil \nu \rceil, \nu > 0$.

Definition 2 [22] The constant x_0 is an equilibrium of fractional-order systems $D^\nu x = f(x,t), x \in \mathbb{R}^n$, if and only if $f(x_0, t) = D^\nu x_0$. Without loss of generality, let the equilibrium be $x_0 = 0$.

Definition 3 [22] A continuous function $\gamma : [0, t) \to [0, \infty)$ is said to be the K-class function if it is strictly increasing and $\gamma(0) = 0$.

The stability analysis of fractional-order systems was investigated in [6,21–26]. Fractional Lyapunov stability is shown by the following fractional-order extension of Lyapunov direct method.

Theorem 1 [23] *Let $x(t) = 0$ be the equilibrium point of the fractional-order system $D^\nu x = f(x,t), x \in \mathbb{D} \subset \mathbb{R}^n$, where $\mathbb{D}$ contains the origin. Let fractional Lyapunov function $V(t, x(t)) : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x. If there exist three K-class functions $\gamma_i, i = 1, 2, 3$ such that*

$$\text{(i) } \gamma_1(\|x\|) \le V(t, x(t)) \le \gamma_2(\|x\|), \tag{2}$$

$$\text{(ii) } D^\nu V(t, x(t)) \le -\gamma_3(\|x\|). \tag{3}$$

where $t \ge 0, x \in \mathbb{D}$. Then, the $x(t) = 0$ is asymptotically stable. Moreover, if the conditions hold globally on $\mathbb{D} = \mathbb{R}^n$, the $x(t) = 0$ is globally asymptotically stable.

It should be noted that, Theorem 1 tells us the pseudo-state stable conditions of fractional-order systems. To be specific, the norm symbol $\|\cdot\|$ represents Euclidean norm or one K-class function. It is obvious that they are equivalent for (2) and (3) always.

Lemma 1 [22] *Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function and $D^\nu x(t) \le D^\nu y(t), x(0) = y(0)$, where $0 < \nu < 1$ is the fractional order. Then, $x(t) \le y(t)$.*

To construct fractional Lyapunov function for fractional-order systems, the power law for fractional-order derivative is introduced before.

Lemma 2 [32] *Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then, for any $p = 2^n, n \in \mathbb{N}$, $D^\nu x^p(t) \le p x^{(p-1)}(t) D^\nu x(t)$, where $0 < \nu < 1$ is the fractional order.*

Proof A simple case of $p = 2$ was shown by [26] with respect to Lemma 1. For the complete proof, one can see [31].

Corollary 1 *Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then, for $p = 2$, $\frac{1}{2} D^\nu x^2(t) \le x(t) D^\nu x(t)$, where $0 < \nu < 1$ is the fractional order.*

Corollary 2 *Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ be a real continuous and differentiable vector function. Then, $D^\nu[x(t)^T P x(t)] \le 2x(t)^T P D^\nu x(t)$, where $0 < \nu < 1$ is the fractional order and $P = diag[p_1, \ldots, p_n] > 0$.*

It will be demonstrated in Sect. 3 that $\frac{1}{2} x(t)^T P x(t)$ (or $P = I$) is always a reasonable fractional Lyapunov function for fractional-order systems.

The concept of adaptive control fractional Lyapunov function (ACFLF) is introduced to test whether an uncertain fractional-order system is pseudo-state feedback stabilized by applying the adaptive control law.

Definition 4 A smooth function $V(t, x(t), \tilde{\theta}) : [0, \infty) \times \mathbb{D} \times \mathbb{R}^m$ is called a ACFLF for $D^\nu x(t) = f(x, u, \theta)$, $f(0, 0, \cdot) = 0$ with the adaptive control law $u = \alpha(x, \hat{\theta})$ if there exist three K-class functions $\gamma_i, i = 1, 2, 3$ such that

$$\text{(i) } \gamma_1(\|\phi\|) \le V(t, x(t), \tilde{\theta}) \le \gamma_2(\|\phi\|), \tag{4}$$

$$\text{(ii) } D^\nu V(t, x(t), \tilde{\theta}) \le -\gamma_3(\|\phi\|). \tag{5}$$

where $t \ge 0, x \in \mathbb{D}$ and $\phi = [x^\top, \theta^\top]^\top, \theta \in \mathbb{R}^m$ is the unknown parameter, the parameter estimate error is $\tilde{\theta} = \theta - \hat{\theta}$ and $D^\nu \hat{\theta} = \tau(x, \hat{\theta})$ is the adaptive law of the parameter estimate. Moreover, if $\mathbb{D} = \mathbb{R}^n$, the ACFLF holds globally.

As the adaptive parameters appear in V, the fractional Lyapunov function is called adaptive control fractional Lyapunov function (ACFLF). The aim of pseudo-state stabilizing uncertain fractional-order nonlinear system is to design an adaptive pseudo-state feedback control law $u = \alpha(x, \hat{\theta})$, $D^\nu \hat{\theta} = \tau(x, \hat{\theta})$ such that the closed-loop system is (globally) asymptotically stable. Actually, finding α, τ and V satisfying (4) and (5) is a difficult task in most cases [21–23,25,26].

Remark 1 The above conditions are sufficient conditions of pseudo-state stabilizability of a class of uncertain fractional-order nonlinear systems. It is possible that there exist other better candidate fractional Lyapunov functions, which may contradict with Theorem 1. However, they are always valid for some specific fractional-order nonlinear systems.

Before Sect. 3, we review the adaptive fractional-order backstepping in an example. For the details, one can refer to [21,32]. By the way, the pseudo-state stabilization problem of fractional-order nonlinear systems with additive disturbance is introduced.

Example 1 Consider a fractional-order nonlinear planar system

$$\begin{cases} D^\nu x = x\xi + x\theta \\ D^\nu \xi = u + \mathrm{d}(t) \end{cases}, \tag{6}$$

where $x, \xi \in \mathbb{R}$ are the states and $u \in \mathbb{R}$ is the control input. θ is the unknown constant parameter. $\mathrm{d}(t)$ is the unknown bounded disturbance, but we do not know its bound.

Two cases (i),(ii) are considered here.

Case (i). When $\mathrm{d}(t) = 0$, let $z_1 = x$ and ξ viewed as the virtual control, the error $z_2 = \xi - \alpha(x, \hat{\theta})$, we have

$$D^\nu z_1 = z_1 \left[z_2 + \alpha(x, \hat{\theta}) \right] + z_1 \theta. \tag{7}$$

Let the parameter estimate error $\tilde{\theta} = \theta - \hat{\theta}$, the first fractional Lyapunov function $V_1(z_1, \hat{\theta}) = \frac{1}{2} z_1^2 + \frac{1}{2\rho}\tilde{\theta}^2$. If choose $\alpha(x, \hat{\theta}) = -C_1 - \hat{\theta}$, we have

$$D^\nu V_1 \le -C_1 z_1^2 + z_1^2 z_2 + \hat{\theta} \left[z_1^2 + \frac{1}{\rho} D^\nu \hat{\theta} \right]. \tag{8}$$

We postpone the choice of update law for $\hat{\theta}$ until the last step. To design the adaptive control u, consider the ACFLF $V_\mathrm{a}(x, \xi, \hat{\theta}) = V_1 + \frac{1}{2} z_2^2$, we have

$$\begin{aligned} D^\nu V_\mathrm{a} \le & -C_1 z_1^2 + \tilde{\theta} \left[z_1^2 - \frac{1}{\rho} D^\nu \hat{\theta} \right] \\ & + z_2 \left[u + z_1^2 - D^\nu \alpha \right]. \end{aligned} \tag{9}$$

One control input and the adaptive law can be chosen by

$$u = -C_2 z_2 - z_1^2 + D^\nu \alpha, \tag{10}$$

$$D^\nu \hat{\theta} = \rho z_1^2. \tag{11}$$

Case (ii). When the disturbance is bounded by a known upper bound $\|\mathrm{d}(t)\|_\infty \le \sigma$. The first fractional Lyapunov design is the same with case (i). Considering the same ACFLF, one control and the adaptive law can be chosen by

$$u = -C_2 z_2 - z_1^2 + D^\nu \alpha - \mathrm{sign}(z_2)\sigma, \tag{12}$$

$$D^\nu \hat{\theta} = \rho z_1^2. \tag{13}$$

where sign is the sign function.

In two cases (i), (ii), we have $D^\nu V_\mathrm{a} \le -C_1 z_1^2 - C_2 z_2^2$. Unless $z = 0$, we have $D^\nu V_\mathrm{a} < 0$. There exists a K-class function γ such that $D^\nu V_\mathrm{a} \le -\gamma(\|\bar{z}\|)$, $\bar{z} = [z^\top, \tilde{\theta}]^\top$. According to Theorem 1, the closed-loop systems are globally asymptotically stable.

Remark 2 Different from the backstepping design [30], the adaptive fractional-order backstepping scheme introduced in Example 1 assumes that the unknown parameters ant their estimates all are unknown constants. In this way, the chain rule for fractional derivative is avoided and the tuning function design become more simple as (11) and (13). The efficiency of this assumption can be proved by the convergence of the resulting closed-loop control systems by use the Theorem 1.

Remark 3 In Example 1, the adaptive fractional-order backstepping can deal with parameter uncertainties by including the parameter estimation scheme. The number of the parameter estimates is equal to that of the unknown parameters. However, the disturbance can be encountered in many physical systems including fractional-order systems. If the upper bound of the disturbance can be determined, the switching control law (12) can be obtained. However, the knowledge of the disturbance may be very limited in reality, where the upper bound is not necessary to be known.

3 Main result

We consider the parametric strict-feedback form of fractional-order nonlinear system with additive disturbance, where the unknown constant parameters appear linearly.

$$\begin{cases} D^{\nu} x_1 = x_2 + \varphi_1^{\top}(x_1)\theta \\ D^{\nu} x_2 = x_3 + \varphi_2^{\top}(x_1, x_2)\theta \\ \vdots \\ D^{\nu} x_{n-1} = x_n + \varphi_{n-1}^{\top}(x_1, \ldots, x_{n-1})\theta \\ D^{\nu} x_n = \beta(x)u + \varphi_n^{\top}(x)\theta + \mathrm{d}(t) \end{cases}, \tag{14}$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ is the unknown constant, $\varphi_i \in \mathbb{R}^m, i = 1, \ldots, m$ are known smooth nonlinear functions, $\mathrm{d}(t)$ is the bounded additive disturbance, and $u \in \mathbb{R}$ is the control input.

The following three assumptions are given firstly.

(i) All pseudo-states of fractional-order system are observable.
(ii) All signals and their first n fractional-order derivatives with fractional order ν are known and bounded.
(iii) The matched disturbance $\mathrm{d}(t)$ is bounded by $\|\mathrm{d}(t)\|_{\infty} \leq \sigma$, where σ is unknown.

Remark 4 When $\mathrm{d}(t) = 0$, the system is the standard parametric strict-feedback form of fractional-order nonlinear system. Assumption 2 is necessary for the boundness of parameter estimates. The disturbance is only assumed to be bounded in Assumption 3, which corresponds most practical cases.

Theorem 2 *Let the parametric strict-feedback form of fractional-order nonlinear system* (14). *If the ACFLF is taken by*

$$V_{\mathrm{a}}(z_1, \ldots, z_n, \tilde{\theta}, \hat{\sigma}) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 + \frac{1}{2}\tilde{\theta}^{\top}\Gamma^{-1}\tilde{\theta} + \frac{1}{2\gamma}\tilde{\sigma}^2, \tag{15}$$

where $z_1 = x_1, z_i = x_i - \alpha_{i-1}(z_1, \ldots, z_{i-1}, \hat{\theta}), i = 2, \ldots, n, \tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimate error and $\tilde{\sigma} = \sigma - \hat{\sigma}$ is the disturbance bound estimate error, that is, there exists an adaptive pseudo-state feedback control u which renders the system globally asymptotically stable. The adaptive pseudo-state feedback control law can be chosen by

$$u = -\frac{1}{\beta(x)}\left[C_n z_n + z_{n-1} + \varphi_n^{\top}(x)\hat{\theta} - D^{\nu}\alpha_{n-1}\left(z_1, \ldots, z_{n-1}, \hat{\theta}\right) + \frac{z_n\hat{\sigma}^2}{|z_n|\hat{\sigma} + C_n z_n^2}\right], \tag{16}$$

$$D^{\nu}\hat{\theta} = \Gamma\sum_{i=1}^{n}\varphi_i(x_1, \ldots, x_i)z_i, \tag{17}$$

$$D^{\nu}\hat{\sigma} = \gamma|z_n|, \tag{18}$$

$$\alpha_{i-1} = -C_{i-1}z_{i-1} - z_{i-2} - \varphi_{i-1}^{\top}(x_1, \ldots, x_{i-1})\hat{\theta} + D^{\nu}\alpha_{i-2}, \quad i = 3, \ldots, n. \tag{19}$$

where $\alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^{\top}(x_1)\hat{\theta}$, $C_1, \ldots, C_n > 0$ are constants. The adaptive parameter $\hat{\theta}$ is updated by (17)*, the adaptive disturbance bound $\hat{\sigma}$ is updated by* (18)*, $\Gamma = diag[\rho_1, \ldots, \rho_m] > 0, \gamma > 0$ are the gains of the adaptive law* (17) *and* (18)*, respectively.*

Proof By use of recursion, we have the following steps.

Step 1. Let $z_1 = x_1$ and x_2 viewed as the virtual control, the error $z_2 = x_2 - \alpha_1(z_1, \hat{\theta})$, we have

$$D^{\nu} z_1 = z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^{\top}(x_1\theta). \tag{20}$$

Note $\tilde{\theta} = \theta - \hat{\theta}$, let the first fractional Lyapunov function $V_1(z_1, \tilde{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^{\top}\Gamma^{-1}\tilde{\theta}$, we have

$$D^{\nu} V_1 \leq z_1\left[z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^{\top}(x_1)\hat{\theta}\right] + \tilde{\theta}^{\top}\left(\varphi_1(x_1)z_1 - \Gamma^{-1}D^{\nu}\hat{\theta}\right). \tag{21}$$

If choose $\alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^{\top}(x_1)\hat{\theta}$, z_2 and $\tilde{\theta}$ are to be governde to zeros. Thus, we have

$$D^{\nu} V_1 \leq -C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^{\top}\left(\varphi_1(x_1)z_1 - \Gamma^{-1}D^{\nu}\hat{\theta}\right). \tag{22}$$

Step 2. Let the error $z_3 = x_3 - \alpha_2(z_1, z_2, \hat{\theta})$, we have

$$D^{\nu} z_2 = z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^{\top}(x_1, x_2)\theta - D^{\nu}\alpha_1(z_1, \hat{\theta}). \tag{23}$$

Let the second fractional Lyapunov function $V_2(z_1, z_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2$, we have

$$D^{\nu} V_2 \leq -C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^{\top}\left(\sum_{i=1}^{2}\varphi_i(x_1, \ldots, x_i)z_i - \Gamma^{-1}D^{\nu}\hat{\theta}\right) + z_2\left[z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^{\top}(x_1, x_2)\hat{\theta} - D^{\nu}\alpha_1(z_1, \hat{\theta})\right]. \tag{24}$$

If choose $\alpha_2(z_1, z_2, \hat{\theta}) = -C_2 z_2 - z_1 - \varphi_2^\top(x_1, x_2)\hat{\theta} + D^\nu \alpha_1(z_1, \hat{\theta})$, z_3 and $\tilde{\theta}$ are to be governed to zeros. Thus, we have

$$D^\nu V_2 \le -\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 + \tilde{\theta}^\top \left(\sum_{i=1}^{2} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right). \tag{25}$$

Step 3. Let the error $z_4 = x_4 - \alpha_3(z_1, z_2, z_3, \hat{\theta})$, we have

$$D^\nu z_3 = z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) + \varphi_3^\top(x_1, x_2, x_3)\theta - D^\nu \alpha_2(z_1, z_2, \hat{\theta}). \tag{26}$$

The third fractional Lyapunov fucntion is chosen by $V_3(z_1, z_2, z_3, \hat{\theta}) = V_2 + \frac{1}{2} z_3^2$, we have

$$\begin{aligned} D^\nu V_3 \le & -\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 \\ & + \tilde{\theta}^\top \left(\sum_{i=1}^{3} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right) \\ & + z_3 \Big[z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) \\ & + \varphi_3^\top(x_1, x_2, x_3)\hat{\theta} - D^\nu \alpha_2(z_1, z_2, \hat{\theta}) \Big]. \end{aligned} \tag{27}$$

If choose $\alpha_3(z_1, z_2, z_3 \hat{\theta}) = -C_3 z_3 - z_2 - \varphi_3^\top(x_1, x_2, x_3)\hat{\theta} + D^\nu \alpha_2(z_1, z_2, \hat{\theta})$, z_4 and $\tilde{\theta}$ are to be governed to zeros. Thus, we have

$$D^\nu V_3 \le -\sum_{i=1}^{3} C_i z_i^2 + z_3 z_4 + \tilde{\theta}^\top \left(\sum_{i=1}^{3} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right). \tag{28}$$

Step $n-1$. Let the error $z_n = x_n - \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})$, we have

$$\begin{aligned} D^\nu z_{n-1} = & \ z_n + \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) \\ & + \varphi_{n-1}^\top(x_1, \ldots, x_{n-1})\theta \\ & - D^\nu \alpha_{n-2}(z_1, \ldots, z_{n-1}, \hat{\theta}). \end{aligned} \tag{29}$$

Let the $n-1$th fractional Lyapunov function $V_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) = V_{n-2} + \frac{1}{2} z_{n-1}^2$, we have

$$\begin{aligned} D^\nu V_{n-1} \le & -\sum_{i=1}^{n-2} C_i z_i^2 + z_{n-2} z_{n-1} \\ & + \tilde{\theta}^\top \left(\sum_{i=1}^{n-1} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right) \\ & + z_{n-1} \Big[z_n + \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) \\ & + \varphi_{n-1}^\top(x_1, \ldots, x_{n-1})\hat{\theta} \\ & - D^\nu \alpha_{n-2}(z_1, \ldots, z_{n-2}, \hat{\theta}) \Big]. \end{aligned} \tag{30}$$

If choose $\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) = -C_{n-1} z_{n-1} - z_{n-2} - \varphi_{n-1}^\top(x_1, \ldots, x_{n-1})\hat{\theta} + D^\nu \alpha_{n-2}(z_1, \ldots, z_{n-2}, \hat{\theta})$, z_n and $\tilde{\theta}$ are to be governed to zeros. Thus, we have

$$D^\nu V_{n-1} \le -\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1} z_n + \tilde{\theta}^\top \left(\sum_{i=1}^{n-1} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right). \tag{31}$$

Step n. The last equation can be transformed into

$$\begin{aligned} D^\nu z_n = & \ \beta(x) u + \varphi_n^\top(x)\theta + \mathrm{d}(t) \\ & - D^\nu \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}). \end{aligned} \tag{32}$$

Regard the unknown upper bound of the disturbance as the unknown parameter, let the ACFLF $V_{\mathrm{a}}(z_1, \ldots, z_n, \hat{\theta}) = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2\gamma}\tilde{\sigma}$, we have

$$\begin{aligned} D^\nu V_{\mathrm{a}} \le & -\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1} z_n \\ & + \tilde{\theta}^\top \left(\sum_{i=1}^{n} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^\nu \hat{\theta} \right) \\ & + z_n \Big[\beta(x) u + \varphi_n^\top(x)\hat{\theta} \\ & - D^\nu \alpha_{n-1}(z_1, \ldots, z_{n-1})\hat{\theta} \Big] + |z_n| \sigma \\ & - \frac{1}{\gamma} \tilde{\sigma} D^\nu \hat{\sigma}. \end{aligned} \tag{33}$$

One control and the adaptive law can be chosen by (16), (17) and (18). Thus, we have

$$
\begin{aligned}
D^{\nu}V_{\mathrm{a}} &\leq -\sum_{i=1}^{n} C_i z_i^2 - \frac{z_n^2\hat{\sigma}^2}{|z_n|\hat{\sigma} + C_n z_n^2} + |z_n|\hat{\sigma} \\
&\leq -\sum_{i=1}^{n-1} C_i z_i^2. \qquad (34)
\end{aligned}
$$

Denote

$\vartheta = [z_1, \ldots, z_n, \tilde{\theta}_1, \ldots, \tilde{\theta}_m, \tilde{\sigma}]^{\top}$;
$\tau = \max\{1, \rho_i, \gamma, i = 1, \ldots, m\}$;
$\varepsilon = \min\{1, \rho_i, \gamma, i = 1, \ldots, m\}$.

Thus, we have

$$
\frac{1}{2\tau}\|\vartheta\|^2 \leq V_{\mathrm{a}}(\vartheta) \leq \frac{1}{2\varepsilon}\|\vartheta\|^2. \qquad (35)
$$

On the other hand, we consider two cases (i) and (ii):

(i) When $[z_1, \ldots, z_{n-1}] \neq 0$, we know $D^{\nu}V_{\mathrm{a}} < 0$. There exists a K-class function γ_1 such that $D^{\nu}V_{\mathrm{a}} \leq -\gamma_1(\|\vartheta\|)$;

(ii) When $[z_1, \ldots, z_{n-1}] = 0$, we know $D^{\nu}V_{\mathrm{a}} \leq 0$. If $D^{\nu}V_{\mathrm{a}} < 0$, similar to case (i), there exists a K-class function γ_2 such that $D^{\nu}V_{\mathrm{a}} \leq -\gamma_2(\|\vartheta\|)$. But, for the case $D^{\nu}V_{\mathrm{a}} = 0$, we know $D^{\nu}V_{\mathrm{a}} = D^{\nu}C \Longrightarrow V_{\mathrm{a}} = C$, where $C = V_{\mathrm{a}}(t = 0)$ is a non-negative constant. So, we know that $\|\vartheta\| = C'$, where C' is a constant only related to Γ, γ. Besides, $C = 0$ if and only if $z = \mathbf{0}, \tilde{\theta} = \mathbf{0}, \tilde{\sigma} = 0$. Therefore, there exists a K-class function γ_3 such that $D^{\nu}V_{\mathrm{a}} \leq -\gamma_3(\|\vartheta\|)$.

With respect to Theorem 1, for the cases (i) and (ii), the pseudo-states in the closed-loop system are asymptotically stable. Besides, the ACFLF (15) holds globally.

So far, this proof is completed. □

Remark 5 It is shown that the parameters of the designed control law are not related to the additive disturbance and unknown system parameters. The unknown upper bound of the additive disturbance is estimated by the designed adaptive law. Besides the estimate of the upper bound of the disturbance, the number of the system parameter estimates is equal to that of the unknown system parameters.

Corollary 3 *If the ACFLF is taken by* (15) *and choose the adaptive control law* (16), (17) *and* (18), *the pseudo-state trajectories of the fractional-order nonlinear system* (14) *will approach to* $x^e = [x_1^e, x_2^e, \ldots, x_n^e]^{\top}$ *asymptotically, where* $x_1^e = 0, x_2^e = -\varphi_1^{\top}(0)\theta, x_{i+1}^e = -\varphi_i^{\top}(0, x_1^e, \ldots, x_{i-1}^e)\theta, i = 2, \ldots, n-1$.

Proof By use of (34) and (35), we have

$$
D^{\nu}V_{\mathrm{a}}(\vartheta) \leq -2c\varepsilon V_{\mathrm{a}}(\vartheta). \qquad (36)
$$

The reason for (36) is that, the existing $\gamma_i, i = 1, 2, 3$ can guarantee there exists a positive constant c such that $V_{\mathrm{a}}(\vartheta) \leq -c\|\vartheta\|^2$ using the discussion of (i) and (ii) in the proof of Theorem 2.

Therefore, there exists a nonnegative function $N(t)$ satisfying $D^{\nu}V_{\mathrm{a}}(\vartheta) + N(t) = -2c\varepsilon V_{\mathrm{a}}(\vartheta)$. Take the Laplace transform, we have

$$
V_{\mathrm{a}}(s) = \frac{V_{\mathrm{a}}(0)s^{\nu-1} - N(s)}{s^{\nu} + 2c\varepsilon}, \qquad (37)
$$

where $V_{\mathrm{a}}(s) = \mathscr{L}[V_{\mathrm{a}}]$ and $V_{\mathrm{a}}(0)$ is a nonnegative constant.

If $\vartheta(0) = 0$, namely $V_{\mathrm{a}}(0) = 0$, the solution is $\vartheta = 0$; if $\vartheta(0) > 0, V_{\mathrm{a}}(0) > 0$. Because $V_{\mathrm{a}}(\vartheta)$ is locally Lipschitz with respect to ϑ, it follows from the existence and uniqueness solution [19], we have

$$
\begin{aligned}
V_{\mathrm{a}}(t) = {} & V_{\mathrm{a}}(0)E_{\nu}(-2c\varepsilon t^{\nu}) - N(t) \\
& \times [t^{\nu-1}E_{\nu,\nu}(-2c\varepsilon t^{\nu})], \qquad (38)
\end{aligned}
$$

where $E_{\nu}, E_{\nu,\nu}$ are Mittag-Leffler type functions [19] and $*$ is convolution operator.

Thus, we have $V_{\mathrm{a}}(t) \leq V_{\mathrm{a}}(0)E_{\nu}(-2c\varepsilon t^{\nu})$. With respect to (35), we have

$$
\|\vartheta\| \leq \sqrt{2\tau V_{\mathrm{a}}(0)E_{\nu}(-2c\varepsilon t^{\nu})}, \qquad (39)
$$

where $\tau V_{\mathrm{a}}(0) > 0$ for $\vartheta(0) \neq 0$.

Therefore, $\lim_{t\to\infty} z_1 = \lim_{t\to\infty} x_1 = \lim_{t\to\infty} z_2 = \cdots = \lim_{t\to\infty} z_n = 0$. By use of recursion, the proof is completed. □

Remark 6 In Corollary 3, it is obvious that the pseudo-state trajectories $x_i, i = 1, \ldots, n$ and $z_i, i = 1, \ldots, n$ are bounded. Besides, the $\alpha_i, i = 1, \ldots, n-1$ are bounded. As the rights of (17) and (18) are Lipschitz with respect to z_i, the parameter estimates $\hat{\theta}, \hat{\sigma}$ are bounded according to Theorem 3 in [19]. The control law (16) is bounded. It can be seen that $u \to -\frac{1}{\beta(x^e)}\varphi_n^{\top}(x^e)\theta$ as $t \to \infty$ by use of (16).

4 Numerical examples

In this section, two examples of fractional-order nonlinear systems are presented to illustrate the effectiveness of the proposed theoretical results. These examples you can find in [4] also. The Grünwald–Letnikov fractional-order difference [4] is used to simulate the fractional-order derivative.

$$D_{t_k}^{\nu} f(t) \approx h^{-\nu} \sum_{i=0}^{k} \mathscr{C}_i^{\nu} f(t_{k-i}), \tag{40}$$

where the time step is set to $h = 0.0005$. $t_k = kh$ is the discrete point and $(-1)^i \mathscr{C}_i^{\nu}$, $i = 1, 2, \ldots$ are binomial coefficients. The fractional Adams method [33] is used to simulate the fractional-order nonlinear systems. In the simulations, we abandon the short memory principle for improving numerical accuracy.

Example 2 Consider the fractional-order Genesio–Tesi system with control u and disturbance d.

$$\begin{cases} D^{\nu} x = y \\ D^{\nu} y = z \\ D^{\nu} z = -\beta_1 x - \beta_2 y - \beta_3 z + \beta_4 x^2 + u + \mathrm{d}(t) \end{cases}, \tag{41}$$

where the fractional order is $\nu = 0.7$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are viewed as unknown constants, which may be caused by modeling uncertainties. The additive disturbance is $\mathrm{d}(t) = x \cos \pi t + 0.1 \sin(3t)$ and $\|d\|_{\infty} \leq \sigma$.

Step 1. Let $z_1 = x$, view y as the virtual control and $z_2 = y - \alpha_1$, we have $D^{\nu} z_1 = z_2 + \alpha_1(z_1)$. Let the first fractional Lyapunov function $V_1 = \frac{1}{2} z_1^2$.

If choose $\alpha_1(z_1) = -K_1 z_1$, $K_1 > 0$, we have $D^{\nu} V_1 \leq -K_1 z_1^2 + z_1 z_2$.

Step 2. Let $z_3 = z - \alpha_2$, we have $D^{\nu} z_2 = z_3 + \alpha_2 + D^{\nu} \alpha_1$. Let the second fractional Lyapunov function $V_2 = V_1 + \frac{1}{2} z_2^2$.

If choose $\alpha_2(z_1, z_2) = -K_2 z_2 - z_1 + D^{\nu} \alpha_1$, $K_2 > 0$, we have $D^{\nu} V_2 \leq -K_1 z_1^2 - K_2 z_2^2 + z_2 z_3$.

Step 3. With the last equation, let the ACFLF

$$V_a = V_2 + \frac{1}{2} z_3^2 + \sum_{i=1}^{4} \frac{1}{2\rho_i} \tilde{\beta}_i^2 + \frac{1}{2\gamma} \tilde{\sigma}^2. \tag{42}$$

where $\tilde{\beta}_i = \beta_i - \hat{\beta}_i$, $\quad i = 1, 2, 3, 4$, $\quad \tilde{\sigma} = \sigma - \hat{\sigma}$.

The adaptive control law can be chosen by

$$\begin{aligned} u = &-K_3 z_3 - z_2 + \hat{\beta}_1 x + \hat{\beta}_2 y + \hat{\beta}_3 z \\ &- \hat{\beta}_4 x^2 + D^{\nu} \alpha_2 - \frac{z_3 \hat{\sigma}^2}{|z_3| \hat{\sigma} + K_3 z_3^2}, \end{aligned} \tag{43}$$

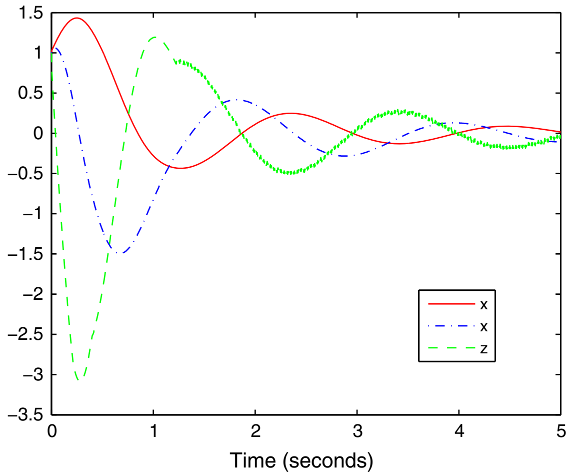

Fig. 1 The state trajectories in Example 2

$$\begin{aligned} &D^{\nu} \hat{\beta}_1 = -\rho_1 x z_3, \quad D^{\nu} \hat{\beta}_2 = -\rho_2 y z_3, \\ &D^{\nu} \hat{\beta}_3 = -\rho_3 z z_3, \quad D^{\nu} \hat{\beta}_4 = \rho_4 x^2 z_3, \quad D^{\nu} \hat{\sigma} = \gamma |z_3|. \end{aligned} \tag{44}$$

Hence, we have $D^{\nu} V_a \leq -K_1 z_1^2 - K_2 z_2^2$.

In the simulation, $K_1 = 6$, $K_2 = 7$, $K_3 = 6$, $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \gamma = 1$. The initial state is $(1, 1, 1)$ and the initial parameter estimates are zeros. The unknown parameters are set to $\beta_1 = \beta_2 = 1.1$, $\beta_3 = 0.45$, $\beta_4 = 1$. The pseudo-state trajectories of the controlled system are shown in Fig. 1. By applying the adaptive control, it is seen that the system converges in a finite time. The control input is shown in Fig. 2. The parameter estimates are shown in Fig. 3. The upper bound estimate is shown in Fig. 4.

Example 3 Consider the fractional-order gyroscope with control u and disturbance d.

$$\begin{cases} D^{\nu} x_1 = x_2 \\ D^{\nu} x_2 = -p(t) x_1 - c_1 x_2 - c_2 x_2^3 + q(t) x_1^3 + u \\ \qquad\quad + \mathrm{d}(t) \end{cases}, \tag{45}$$

where $p(t) = \frac{\alpha^2}{4} - f \sin(\omega t)$, $q(t) = \frac{\alpha^2}{12} - \frac{\beta}{6} - \frac{f \sin(\omega t)}{6}$, $\alpha^2 = 100$, $\beta = 1$, $\omega = 25$, $f = 35.5$, the fractional order is $\nu = 0.7$ and c_1, c_2 are viewed as unknown constants, which may be caused by modeling uncertainties. The additive disturbance is $\mathrm{d}(t) = 0.5 \cos \pi t + 0.1 \sin(3t)$ and $\|d\|_{\infty} \leq \sigma$.

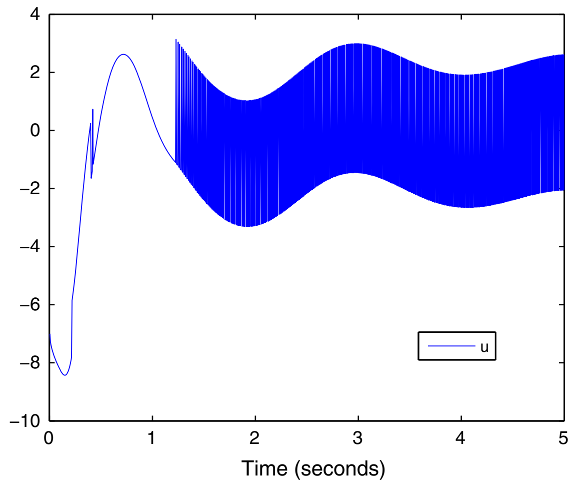

Fig. 2 The control input in Example 2

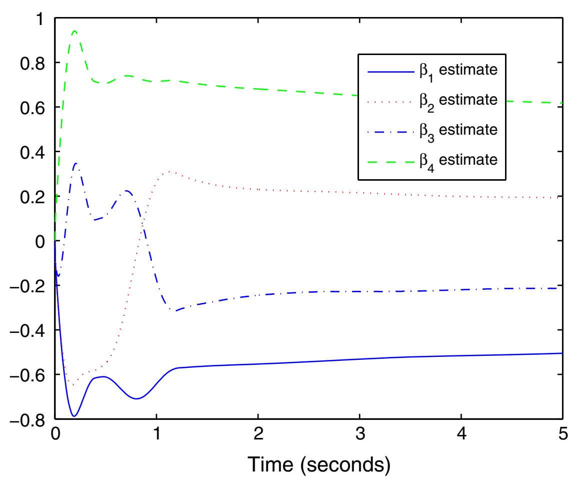

Fig. 3 The parameter estimates in Example 2

Step 1. Let $z_1 = x_1$, view x_2 as the virtual control and $z_2 = x_2 - \alpha_1$, we have $D^\nu z_1 = z_2 + \alpha_1(z_1, \hat{c}_1, \hat{c}_2)$.

Denote $\tilde{c}_1 = c_1 - \hat{c}_1, \tilde{c}_2 = c_2 - \hat{c}_2$. Let the first fractional Lyapunov function $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\rho_1}\tilde{c}_1^2 + \frac{1}{2\rho_2}\tilde{c}_2^2$.

If choose $\alpha_1(z_1, \hat{c}_1, \hat{c}_2) = -K_1 x_1$, $K_1 > 0$, we have

$$D^\nu V_1 \leq -K_1 z_1^2 + z_1 z_2 - \frac{1}{\rho_1}\tilde{c}_1 D^\nu \hat{c}_1 - \frac{1}{\rho_2}\tilde{c}_2 D^\nu \hat{c}_2.$$

Step 2. With $D^\nu z_2 = -p(t)x_1 - c_1 x_2 - c_2 x_2^3 + q(t)x_1^3 + u + \mathrm{d}(t) - D^\nu \alpha_1$, note $\tilde{\sigma} = \sigma - \hat{\sigma}$ let the candidate ACFLF $V_\mathrm{a} = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\sigma}^2$.

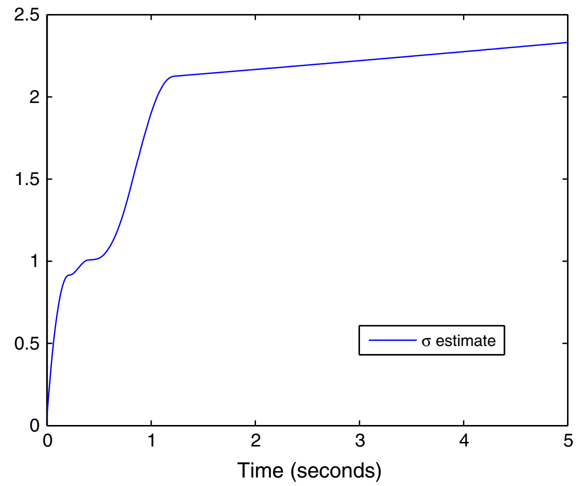

Fig. 4 The upper bound estimate in Example 2

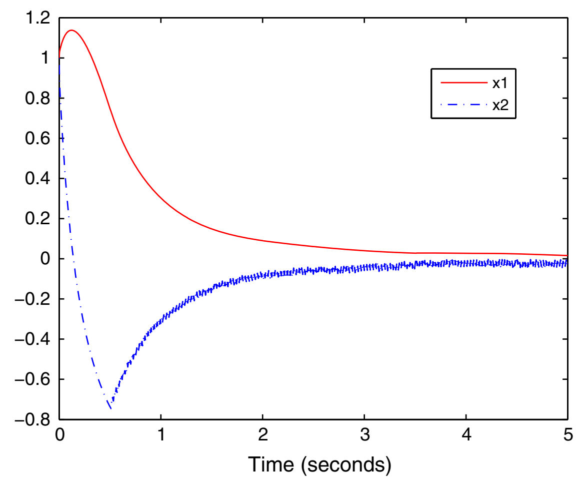

Fig. 5 The state trajectories in Example 3

The adaptive control law can be chosen by

$$u = -K_2 z_2 - z_1 + p(t)x_1 + \hat{c}_1 x_2 + \hat{c}_2 x_2^3 - q(t)x_1^3 - K_1 x_2 - \frac{z_2\hat{\sigma}^2}{|z_2|\hat{\sigma} + K_2 z_2^2}, \tag{46}$$

$$D^\nu \hat{c}_1 = -\rho_1 x_2 z_2, \quad D^\nu \hat{c}_2 = -\rho_2 x_2^3 z_2,$$
$$D^\nu \hat{\sigma} = \gamma |z_2|. \tag{47}$$

Hence, we have $D^\nu V_\mathrm{a} \leq -K_1 z_1^2$.

In the simulation, $K_1 = 6, K_2 = 7, \rho_1 = 3, \rho_2 = 4, \gamma = 2$. The initial state is $(1, 1)$ and the initial parameter estimates are zeros. The unknown parameters are set to $c_1 = 0.5, c_2 = 0.05$. The pseudo-

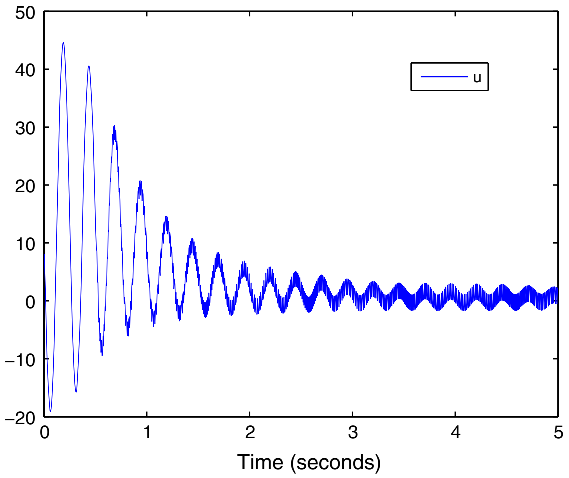

Fig. 6 The control input in Example 3

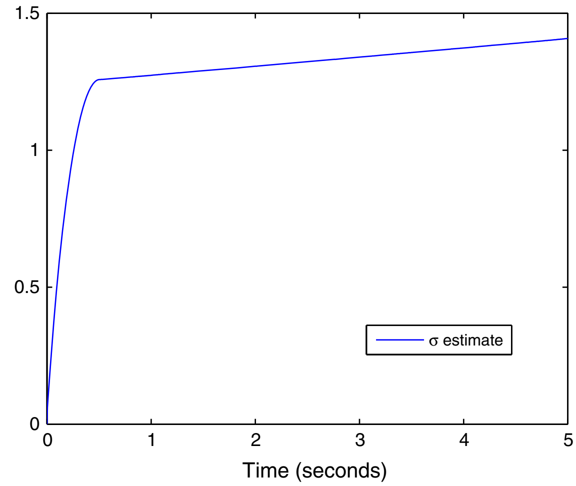

Fig. 8 The upper bound estimate in Example 3

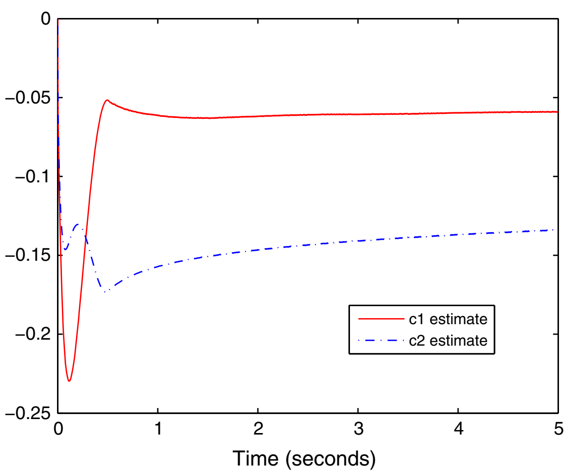

Fig. 7 The parameter estimates in Example 3

state trajectories of the controlled system are shown in Fig. 5. By applying the adaptive control, it is seen that the system converges in a finite time. The control input is shown in Fig. 6. The parameter estimates are shown in Fig. 7. The upper bound estimate is shown in Fig. 8.

In Examples 2 and 3, it is shown that the proposed fractional-order controller can stabilize the pseudo-states of a class of fractional-order nonlinear systems effectively. In the presence of additive disturbance, the system pseudo-states, parameter estimates, and control inputs are all bounded.

5 Conclusions

In this paper, we present an adaptive fractional-order backstepping control design for a class of fractional-order nonlinear systems with additive disturbance. The proposed control laws do not require the specific knowledge on the disturbance and the system parameters. The asymptotic pseudo-state stability of the closed-loop system is guaranteed in terms of fractional Lyapunov stability. Simulation results are provided to illustrate the effectiveness of the control scheme.

The future work can be directed to investigate the adaptive fractional-order backstepping control design for general uncertain fractional-order nonlinear systems. Besides, the relationship between the Lyapunov stability and the fractional Lyapunov stability should be bridged further.

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